Game Theory - Week 6

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Overview

- Coalitional Game Theory: Definitions
- The Shapley Value
- The Core
- Comparing the Core and Shapley Value in an Example

Introduction

- Our focus is on what **groups of agents**, rather than individual agents, can achieve.
- Given a set of agents, a coalitional game defines how well each group (or coalition) of agents can do for itself.
- We are **not** concerned with:
  - how the agents make individual choices within a coalition;
  - how they coordinate;
- ...instead, we take the payoffs to a coalition as given.

Definition

- Transferable utility assumption:
  - payoffs may be redistributed among a coalition's members.
  - satisfied whenever payoffs are dispensed in a universal currency.
  - each coalition can be assigned a single value as its payoff.

Definition (Coalitional game with transferable utility)

A coalitional game with transferable utility is a pair (N, v), where

- N is a finite set of players, indexed by i ; and
- $v : 2^N \to \mathbb{R}$ associates with each coalition $S \subseteq N$ a real-valued payoff $v(S)$ that the coalition's members can distribute among themselves. We assume that $v(\emptyset) = 0$.

Using Coalitional Game Theory

Questions we use coalitional game theory to answer:

1. **Which coalition** will form?
2. How should that coalition **divide its payoff** among its members?

The answer to (1) is often “the grand coalition ” (all agents in N) though this can depend on having made the right choice about (2).

Superadditive games

Definition (Superadditive game)

A game $G = (N, v)$ is superadditive if for all $S, T \subset N$,if $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$.

- Superadditivity is justified when coalitions can always work without interfering with one another
  - the value of two coalitions will be no less than the sum of their individual values.
  - implies that the grand coalition has the highest payoff.

Analyzing coalitional games

1. Which coalition will form?
   - we'll consider cases where the answer is **the grand coalition**.
   - makes sense for superadditive games.

2. How should the coalition divide its payoff?
   - in order to be **fair**.
   - in order to be **stable**.

Coalitional or Cooperative Games

- Question: what is a 'fair' way for a coalition to divide its payoff?
- This depends on how we define 'fairness.'
- One Approach: identify **axioms** that express properties of a fair payoff division.

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  - Cannot pay everyone their marginal Contribution!
- We will have to use some weighting system - how should it be designed?
- Shapley's axioms give us one answer...

Symmetry

- i and j are **interchangeable relative** to v if they always contribute the same amount to every coalition of the other agents.
  - for all S that contains neither i nor $j, v(S \cup \{i\}) = v(S \cup \{j\})$.

Axiom (Symmetry)

For any v, if i and j are interchangeable then $\psi_i(N, v) = \psi_j(N, v)$

- Interchangeable agents should receive the same shares/payments.

Dummy Players

- i is a **dummy player** if the amount that i contributes to any coalition is 0.
  - for all $S : v(S \cup \{i\}) = v(S)$.

Axiom (Dummy player)

For any v, if i is a dummy player then $\psi_i(N, v) = 0$.

- Dummy players should receive nothing.

Additivity

- If we can separate a game into two parts $v = v_1 + v_2$, then we should be able to decompose the payments:

Axiom (Additivity)

For any two v_1 *and* $, v_2, \psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2\ )$ *for each* i*, where the game* $(N, v_1 + v_2)$ *is defined by* $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ *for every coalition* S*.*

Shapley Value

Given a coalitional game (N, v), the **Shapley Value** divides payoffs among players according to:

$$\phi_i(N, v) = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)! \Big[v(S \cup \{i\}) - v(S) \Big].$$

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Theorem

Given a coalitional game (N, v), there is a unique payoff division $x(v) = \phi(N, v)$ that divides the full payoff of the grand coalition and that satisfies the Symmetry, Dummy player and Additivity axioms: the Shapley Value

Understanding the Shapley Value

$$\phi_i(N, v) = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)! \Big[v(S \cup \{i\}) - v(S) \Big].$$

This captures the "marginal contributions" of agent i, averaging over all the different sequences according to which the grand coalition could be built up.

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- For any such sequence, look at agent i's marginal contribution when added: $[v(S \cup \{i\}) - v(S)]$.

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- Weight this quantity by the $|S|!$ ways the set S could have been formed prior i's addition and by the $(|N| - |S| - 1)!$ ways the remaining players could be added.

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This captures the "marginal contributions" of agent i, averaging over all the different sequences according to which the grand coalition could be built up.

- Sum over all possible sets S and average by dividing by $|N|!$: the number of possible orderings of all the agents.

Two Partners Sharing their Profits:

$$v(\{1\}) = 1, v(\{2\}) = 2, v(\{1,2\}) = 4$$

Shapley Value

- The Shapley Value allocates the value of a group according to marginal contribution calculations.
- Captured by some simple axioms and logic.
- Other axioms and approaches lead to other allocations of value for example the "Core" up next.

Stable payoff division

- The Shapley value defined a **fair way** of dividing the grand coalition's payment among its members.
  - However, this analysis ignored questions of stability.
- Would the agents be willing to form the **grand coalition** given the way it will divide payments, or would some of them prefer to form **smaller coalitions**?
  - Unfortunately, sometimes smaller coalitions can be more attractive for subsets of the agents, even if they lead to lower value overall.

Stable payoff division

Example (Voting game)

A parliament is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a $100 million spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

- Shapley values: (50 , 16.67, 16.67, 16.67).
- Can a subcoalition gain by defecting? While A can't obtain more than 50 on its own, A and B have incentive to defect and divide the $100 million between them (e.g., (75, 25)).

The Core

- Under what payment divisions would the agents **want to form the grand coalition**?
- They would want to do so if and only if the payment profile is drawn from a set called the **core**.

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Definition (Core)

A payoff vector x is in the core of a coalitional game (N, v) iff

$$\forall S \subseteq N, \sum_{i \in S} x_i \geq v(S).$$

- The sum of payoffs to the agents in any subcoalition S is at least as much as they could earn on their own.
- Analogous to **Nash equilibrium**, except that it allows deviations by groups of agents.

Existence and Uniqueness

1. Is the core always **nonempty**?
2. Is the core always **unique**?

Existence and Uniqueness

1. Is the core always **nonempty**?

- Consider again the voting game.
- The set of minimal coalitions that meet the required 51 votes is $\{A, B\}, \{A, C\}, \{A, D\}$, and$\{B, C, D\}$.
- If the sum of the payoffs to parties B, C, and D is less than \$100 million, then this set of agents has incentive to deviate.
- If B, C, and D get the entire payoff of \$100 million, then A will receive \$0 and will have incentive to form a coalition with whichever of B, C, and D obtained the smallest payoff.

2. Is the core always **unique**?

Existence and Uniqueness

1. Is the core always **nonempty**?

2. Is the core always **unique**?

- Consider changing the example so that an 80 % majority is required.
- The minimal winning coalitions are now $\{A, B, C\}$ and $\{A, B, D\}$.
- Any complete distribution of the \$100 million among A and B now belongs to the core.
  - all winning coalitions need the support of these two parties.

Simple Games

Definition (Simple game)

A game $G = (N, v)$ is simple if for all $S \subset N, v(S) \in \{0, 1\}$.

Definition (Veto player)

A player i is a veto player if $v(N \setminus \{i\}) = 0$.

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Theorem

In a simple game the core is empty iff there is no veto player. If there are veto players, the core consists of all payoff vectors in which the nonveto players get 0.

Airport Game

Definition (Airport game)

Several nearby cities need airport capacity, with different cities needing to accommodate aircraft of different sizes. If a new regional airport is built the cities will have to share its cost, which will depend on the largest aircraft that the runway can accommodate. Otherwise each city will have to build its own airport. This situation can be modeled as a coalitional game (N, v), where N is the set of cities, and $v(S)$ is the sum of the costs of building runways for each city in S minus the cost of the largest runway required by any city in S.

Convex games

Definition (Convex game)

A game $G = (N, v)$ is convex if for all $S, T \subset N$,
$v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$.

- Convexity is a stronger condition than superadditivity.
- The Airport game is convex.

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Theorem

Every convex game has a nonempty core.

Theorem

In every convex game, the Shapley value is in the core.

Compare Core and Shapley Value in an Example

- UN security council: 15 members

Compare Core and Shapley Value in an Example

- UN security council: 15 members
  - 5 permanent members: China, France, Russia, UK, US
  - 10 temporary members
  - 5 permanent members can veto resolutions

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- UN security council: represent it as a cooperative game.

Compare Core and Shapley Value in an Example

- UN security council: represent it as a cooperative game.
  - China, France, Russia, UK, US are labeled {1, 2, 3, 4, 5}
  - $v(S)$ if $\{1, 2, 3, 4, 5\} \subset S$ and $\#S \geq 8$,
  - $v(S) = 0$ otherwise.

Compare Core and Shapley Value in an Example

- Let's start with a three-player game that has a similar structure:

Compare Core and Shapley Value in an Example

- Let's start with a three-player game that has a similar structure:
  - 1 permanent member with a veto and 2 temporary members.
  - $v(S) = 1$ if $1 \in S$ and $\#S \geq 2$,
  - $v(S) = 0$ otherwise.

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- $v(S) = 1$ if $1 \in S$ and $\#S \geq 2, v(S) = 0$ otherwise.
  - Core: $x_1 + x_2 \geq 1, x_1 + x_3 \geq 1, x_1 + x_2 + x_3 = 1, x_i \geq 0$.
  - Core: $x_1 = 1, x_2 = 0, x_3 = 0$.

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    - 1's value: $v(\{1,2,3\}) - v(\{2,3\}) = 1$ weighted by $\frac{2}{6}$, $v(\{1,2\}) - v(\{2\}) = 1$ weighted by $\frac{1}{6}$, $v(\{1,3\}) - v(\{3\}) = 1$ weighted by $\frac{1}{6}$

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    - 3's value: $v(\{1,3\}) - v(\{1\}) = 1$ weighted by $\frac{1}{6}$
  - Shapley Value: $x_1 = \frac{2}{3}, x_2 = \frac{1}{6}, x_3 = \frac{1}{6}$.

A way to the Shapley Value:

Cooperative Games

- Model complex multilateral bargaining and coalition formation, without specifying the particulars of a normal or extensive form.
  - Core: Based on coalitional threats - each coalition must get at least what it can generate alone.
  - Shapley Value: based on marginal contributions: what does each player contribute to each possible coalition.
  - Other solutions ...