# Game Theory - Week 4

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# Overview

- The normal form game representation does not incorporate any notion of sequence or time of the action of the players
- The extensive form is an alternative representation that makes the temporal structure explicit
- Two variants:
    - Perfect information
    - Imperfect information

## Definition

A (finite) perfect-information game (in extensive form) is defined by the tuple $(N, A, H, Z, \chi, \rho, \sigma, u)$ where

- Players : $N$
- Actions : $A$
- Choice nodes and labels for these nodes :
  - Choice Nodes : $H$
  - Action Function : $\chi : H \to 2^A$
  - Player Function : $\rho : H \to N$
- Terminal Nodes : $Z$
- Successor Function : $\sigma : H \times A \to H \cup Z$ maps a choice node and an action to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then : $h_1 = h_2 \quad , \quad a_1 = a_2$
  - Choice nodes form a tree : nodes encode history

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- Successor Function : $\sigma : H \times A \to H \cup Z$
- Utility Function : $u = (u_1, \ldots, u_n) : u_i : Z \to \mathbb{R}$ is a utility function for player $i$ on the terminal nodes $Z$

A pure strategy for a player in a perfect-information game is a complete specification of which action to take at each node belonging to that player.

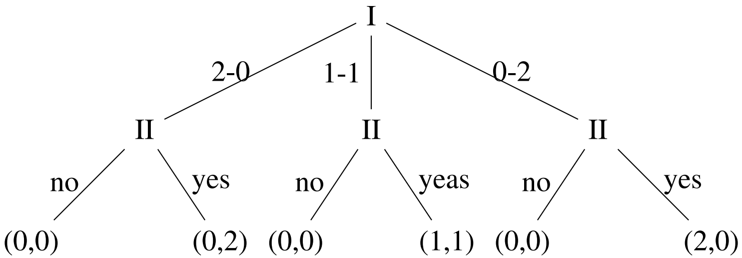
**Definition**

Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect-information extensive-form game. Then the pure strategies of player  consist of the cross product

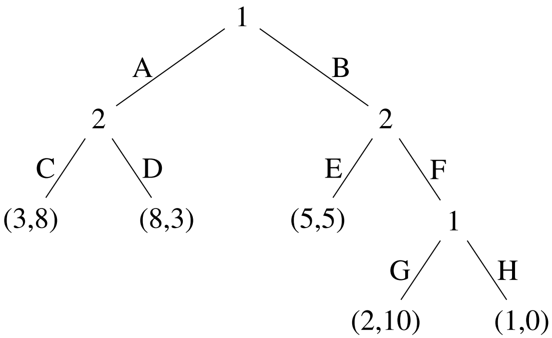
$$\prod_{h \in H, \rho(h)=i} \chi(h)$$

# Example

A brother and sister going to share 2 dollars:



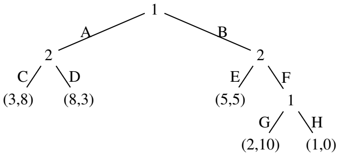
# Example




- How many pure strategies does each player have?
-
    - Player 1 : 4
    - Player 2 : 4

# Induced Normal Form Games

- In fact, the connection to the normal form is even tighter
- We can convert an extensive-form game into normal form

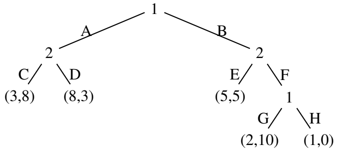



| | CE | CF | DE | DF |
|---|---|---|---|---|
| AG | (3,8) | (3,8) | (8,3) | (8,3) |
| AH | (3,8) | (3,8) | (8,3) | (8,3) |
| BG | (5,5) | (2,10) | (5,5) | (2,10) |
| BH | (5,5) | (1,0) | (5,5) | (1,0) |

- This help us find the Nash equilibrium
- We can't always perform the reverse transformation e.g. matching pennies

**Theorem**

*Every perfect information game in extensive form has a pure strategy Nash equilibrium*




- There's something intuitively wrong the equilibrium $(B, H), (C, E)$
- Why would player 1 ever choose to play H if he got to the second choice node?
- After all, $G$ dominates $H$for him
- He does it to threaten player 2, to prevent him from choosing F, and so gets 5
- However, this seems like a non-credible threat
- If player 1 reached his second decision node, would he really follow through and play $H$?

## Subgame of $G$ rooted at $h$

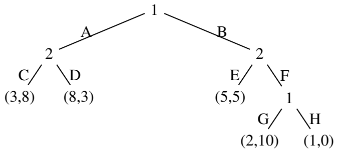
The subgame of $G$ rooted at $h$ is the restriction of $G$ to the descendents of $H$

## Subgame of $G$

The set of subgames of $G$ is defined by the subgames of $G$ rooted at each of the nodes in $G$

- $s$ is a subgame perfect equilibrium of $G$ iff for any subgame $G'$ of $G$, the restriction of $s$ to $G'$ is a Nash equilibrium of $G'$
- Notes :
  - Since $G$ is its own subgame then every subgame perfect equilibrium is a Nash equilibrium
  - This definition rules out ”non credible threats”

# Which equilibria are subgame perfect




- Which equilibria from the example are subgame perfect?
  - $(A,G),(C,F)$
  - $(B,H),(C,E)$
  - $(A,H),(C,F)$

# Computing Subgame Perfect Equilibria

**Theorem**

*Every finite extensive-form game of perfect information has a subgame-perfect pure Nash equilibrium which can be computed by backward induction.*

- Idea: identify the equilibria in the bottom-most trees, and adopt these as one moves up the tree

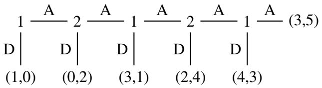
**Algorithm 1** Computing Subgame Perfect Equilibria

**function** BACKWARDINDUCTION(node $h$)
  **if** $h \in Z$ **then**
    **return** $u(h)$
  best_size $\leftarrow -\infty$
  **for** $a \in \chi(h)$ **do**
    util_at_child $\leftarrow$ BACKWARDINDUCTION$(\sigma(h, a))$
    **if** util_at_child$_{\rho(h)}$ $>$ best_util$_{\rho(h)}$ **then**
      best_util $\leftarrow$ util_at_child
  **return** best_util

- *util_at_child* is a vector denoting the utility for each player.
- The procedure doesn't return an equilibrium strategy, but rather labels each node with a vector of real numbers.
  - This labeling can be seen as an extension of the game's utility function to the non-terminal nodes.
  - Equilibrium strategies take a best action at each node.

# Example - Centipede

There is an increasing pot of money. Two players take turns choosing to take a slightly larger share of the pot or to pass the pot to the other player. The payoffs are arranged so that if one passes the pot to one's opponent and the opponent takes the pot on the next round, one receives slightly less than if one had taken the pot on this round, but after an additional switch, the potential payoff will be higher. Therefore, although a player has the incentive to take the pot at each round, it would be better for them to wait.




- What happens when we use this procedure on Centipede?
  - In the only equilibrium, player 1 goes down in the first move
  - This outcome is Pareto-dominated by all but one other outcome
- Two considerations:
  - Practical: human subjects don't go down right away
  - Theoretical: What should player 2 do if player 1 doesn't go down?
    - SPE analysis says to go down. However, that same analysis says that P1 would already have gone down. How should player 2 update beliefs upon observation of a measure zero event?
    - but if player 1 knows that player 2 will do something else, it is rational for him not to go down anymore ... a paradox
    - there's a whole literature on this question

- laboratory experiments have shown that backward inducation equilibrium rarely arises when ”typical” humans play this game
- On the other hand, when the experimental subjects were chess players, the subgame perfect outcome did indeed arise. Perhaps this is because chess players are more adept at backward induction.

# Ultimatum Bargaining

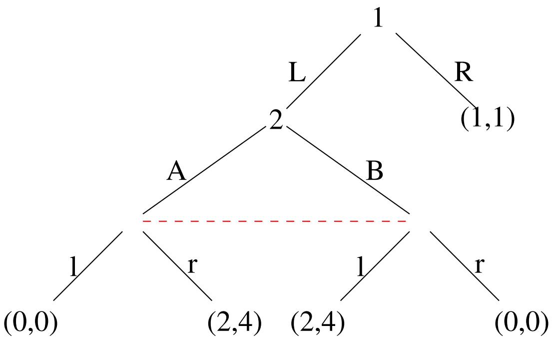
- Player 1 makes an offer $x \in \{1, \ldots, 10\}$ to player 2
- Player 2 can accept or reject
- Player 1 gets $10 - x$ and player 2 gets $x$ if accepted
- Both get 0 if rejected

- So far, we've allowed players to choose an action at every choice node.
  - This implies that players know the node they are in and all the prior choices, including those of other agents.
  - We may want to model agent needing to act with partial or no knowledge of the action taken by others, or even themselves
- Imperfect information extensive-form games:
  - Each player's choice nodes partitioned into information sets
  - Agents cannot distinguish between choice nodes in the same information set

## Imperfect Information Extensive Game

An imperfect-information game (in extensive form) is a tuple $(N, A, H, Z, \chi, \rho, \sigma, u, I)$, where

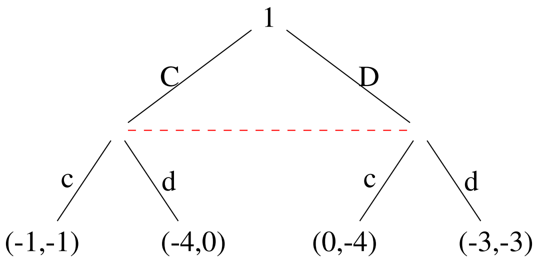
- $(N, A, H, Z, \chi, \rho, \sigma, u)$ is a perfect information extensive form game
- $I = (I_1, \dots, I_n)$ where $I_i = (I_{i,1}, \dots, I_{i,k_i})$ is a partition of $\{h \in H : \rho(h) = i\}$ with the property that $\chi(h) = \chi(h')$ and $\rho(h) = \rho(h')$ whenever there exists a $j$ for which $h \in I_{i,j}$ and $h' \in I_{i,j}$




- What are the equivalence classes for each player?
- How should we define pure strategies for each player?
  - Choice of an action in each information set
- Formally, the pure strategies of player $i$ consist of the cross product $\prod_{I_{i,j} \in I_i} \chi(I_{i,j})$

# Normal Form Games

- We can represent any nomral form game




- It would be the same if put player 2 at the root.

# Randomized Strategies

- There are two meaningfully different kinds of randomized strategies in imperfect information extensive form games
  - Mixed Strategies
  - Behavioral Strategies
- Mixed Strategies : randomize over pure strategies
- Behavioral Strategies : independent coin toss every time an information set is encountered

## Behavioral Strategies

A behavioral strategy $b_i$ for player $i$ in an extensive form game is a map that associates to each information set $I$ of $i$ a probability distribution $b_i(I)$ over the actions that available to $i$ at $I$.

## Realization Equivalence

Two strategies $s_i$ and $s'_i$ for player $i$ in an extensive form game are *realization equivalent* if for each strategy $S_{-i}$ of the opponents and every node $v$ in the game tree, the probability of reaching $v$ when strategy profile $(s_i, S_{-i})$ is employed is the same as the probability of reaching $v$ when $(s'_i, S_{-i})$

## Theorem

*Consider an extensive game of perfect recall. Then for any player i and every mixed strategy $s_i$, there is a realization-equivalent $s'_i$ that is induced by a behavioral strategy $b_i$. Hence for every possible strategy of opponents $S_{-i}$, every player i' expected utility under $(s_i, S_{-i})$ is the same as his expected utility under $(s'_i, S_{-i})$*

**Corollary**

*In a finite extensive game of perfect recall, there is a Nash equilibrium in behavioral strategies.*

$$s_i(A) = \text{Probability that strategy } s_i \text{ put on a set of pure strategies A}$$

Let $v$ and $v_1, \ldots, v_{t-1}$ be player $i$ nodes such $v_1, \ldots, v_{t-1}$ are nodes that appear in the way to node $v$ and let $a_1, \ldots, a_{t-1}$ actions taken by $i$ in each node

$$\Omega(v) = \text{Set of pure strategies of player } i \text{ where he plays } a_j \text{ at } v_j$$

$$\Omega(v, a) = \{s \in \Omega(v) | \text{action } a \text{ is played at } v\}$$

$b_i(v) =$
Conditional distribution over actions at $v$ given that $v$ is reached

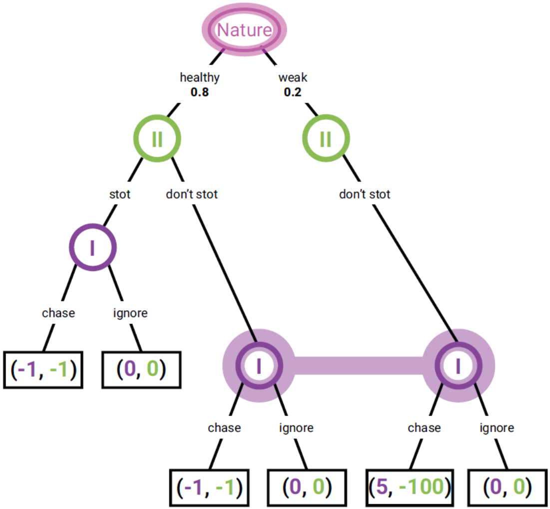
$$b_i(v)_a = \frac{s_i(\Omega(v, a))}{s_i(\Omega(v))}$$

# Example - Lions and Antelopes

Antelopes have been observed to jump energetically when they notice a lion. Why do they expend energy in this way? One theory is that the antelopes are signaling danger to others at some distance, in a community-spirited gesture. However, the antelopes have been observed doing this even when there are no other antelopes nearby. The currently accepted theory is that the signal is intended for the lion, to indicate that the antelope is in good health and is unlikely to be caught in a chase. This is the idea behind signaling.

- One can observe that this is an **incomplete information game**. It means that a player does not know exactly what game he is playing, e.g., how many players there are, which moves are available to the players, and what the payoffs at terminal nodes are.
- Consider the situation of an antelope catching sight of a lioness in the distance. Suppose there are two kinds of antelope, healthy ($H$) and weak ($W$). A lioness can catch a weak antelope but has no chance of catching a healthy antelope (and would expend a lot of energy if he tried).
- This can be modeled as a combination of two simple games ($A^H$ and $A^W$), depending on whether the antelope is healthy or weak, in which case the antelope has only one strategy (to run if chased), but the lioness has the choice of chasing ($C$) or ignoring ($I$).

## Imperfect information games



- The lioness does not know which game she is playing - and if twenty percent of the antelopes are weak, then the lioness can expect a payoff of $0.8 * (-1) + 0.2 * 5 = 0.2$ by chasing. However, the antelope does know, and if a healthy antelope can credibly convey that information to the lioness by jumping very high, both will be better off - the antelope much more than the lioness!